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THESIS

A STUDY OF FOUTZ'S MULTIVARIATE
GOODNESS-OF-FIT TEST

by

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March 1982

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A Study of Foutz's Multivariate Goodness-of-Fit Test

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The empirical power of a new multivariate goodness-of-fit test proposed by Foutz (1980) is investigated. The test has been applied to Monte Carlo samples from bivariate and trivariate normal distributions with a variety of mean vectors and covariance matrices. The null hypothesis tested is that the sample is from a multivariate normal distribution with $\underline{0}$ mean vector and covariance matrix the identity $\underline{I}$. The observed number of rejections in 5000 replications is used as the measure of effectiveness of the test. The results indicate that the Foutz test is quite capable of detecting mean and variance shifts but is not as powerful against covariance shifts.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

I. INTRODUCTION

In statistical analysis, choosing the correct distribution to model available data is of importance. A class of procedures known as goodness-of-fit tests has been derived to test the hypothesis that a set of samples is from a given distribution. Many of these tests are readily available and are well known, such as the Chi-square or the Kolmogorov-Smirnoff (K.S.) goodness-of-fit test. These tests were designed for univariate distributions and are not usable as multivariate goodness-of-fit tests in their present form.

In 1980 Robert V. Foutz [Ref. 1] proposed a new multivariate goodness-of-fit test that will be called the Fn test in the sequel. In analogy to the K.S. test the Fn test compares a hypothesized cumulative distribution function (CDF) with a "continuous empirical distribution function" (CEDF) formed from sampled data. Foutz found the null distribution of the test to be distribution free as well as being independent of the number of variates p.

Foutz obtained an integral expression for the null distribution of the Fn test statistic, and closed form solutions for sample size 2 or 3 were provided. The complexity of the integral expression increases with sample size, and a normal approximation to the null distribution was given for use with larger sample sizes. Although

the Fn test was designed as a multivariate goodness-of-fit test it can also be used to fit univariate distributions. Franke and Jayachandran [Ref. 2] compared the empirical power of the Fn test with that for the Chi-square test and the K.S. test. The results indicated that the Fn test competes well with these other tests.

The power of the Fn test as a multivariate goodness-of-fit test is investigated in this thesis. A description of the Foutz test is given in Section II and the Monte Carlo methods of simulation are presented in Section III. The results and conclusions are in Section IV. A Fortran code for the application of the Fn test is available in the Appendix.

II. THE FOUTZ TEST

The Fn test for multivariate goodness-of-fit is based on a comparison of a hypothesized CDF with a continuous empirical distribution function (CEDF) derived from a sample. The first step in the determination of the CEDF is the construction of what are known as statistically equivalent blocks. A general method for determining statistically equivalent blocks, due to Anderson [Ref. 3], is described below.

Given a random sample $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}$ from a p-variate continuous distribution, select n-1 functions $h_k(\underline{X})$, $k = 1,2,\ldots,$ n-1, not necessarily distinct, such that each $h_k(\underline{X})$ has a continuous distribution. These functions are referred to as cutting functions and will be used to partition the sample space into blocks. Let $k_1, k_2, \ldots, k_{n-1}$ be a permutation of $1,2,\ldots,n-1$. Order the $\underline{X}_i$'s according to $h_{k_1}(\underline{X})$ and define $\underline{X}(k_1)$ as the k_1th order statistic. The sample space is partitioned into two blocks.

$$B_1 = \left\{ \underline{X}: \; h_{k_1}(\underline{X}) \leq h_{k_1}(\underline{X}(k_1)) \right\}$$

$$B_2 = \left\{ \underline{X}: \; h_{k_1}(\underline{X}) > h_{k_1}(\underline{X}(k_1)) \right\} .$$

At the second step if $0 < k_2 < k_1$ the k-1 $\underline{X}$'s in B_1 are ordered according to $h_{k_2}(\underline{X})$; $\underline{X}(k_2)$ is defined as the k_2th in the ordering. Define a cut on B_1 obtaining 3 blocks as follows:

$$B_{11} = B_1 \cap \left\{ \underline{X}: \; h_{k_2}(\underline{X}) \le h_{k_2}(X(k_2)) \right\},$$

$$B_{12} = B_1 \cap \left\{ \underline{X}: \; h_{k_2}(\underline{X}) > h_{k_2}(\underline{X}(k_2)) \right\},$$

$$B_{20} = B_2.$$

Now consider the other alternative, $k_2 > k_1$. We rank the $((n-1)-k_1)$ $\underline{X}$'s in the second block B_2 according to $h_{k_2}(\underline{X})$ and let $\underline{X}(k_2)$ be the (k_2-k_1)th largest in the ranking. Defining a cut at $h_{k_2}(\underline{X}(k_2))$ we obtain the 3 blocks,

$$B_{10} = B_1,$$

$$B_{21} = B_2 \cap \left\{ X: \; h_{k_2}(X) \le h_{k_2}(X(k_2)) \right\},$$

$$B_{22} = B_2 \cap \left\{ X: \; h_{k_2}(X) > h_{k_2}(X(k_2)) \right\}.$$

The process is continued until all the cutting functions are exhausted. This results in a partition of the sample space into n statistically equivalent blocks, which are denoted by B_i, $i = 1,\ldots,n$.

In the univariate case an intuitively appealing choice for the cutting functions is the identity function viz., $h(X) = X$ for all k. The resulting statistically equivalent blocks are then $(-\infty, X(1)], (X(1), X(2)], \ldots, (X(n-1), +\infty)$ where $X(j)$ is the jth order statistic. The multivariate analogue is to choose

individual coordinates as cutting functions, viz., $h_k(\underline{X}) = \underline{X}^{(j)}$, the jth coordinate of $\underline{X}$. An example illustrating the construction of the blocks in the bivariate case is given below for a sample of size 8.

Let (2,4,6,8,1,3,5,7) be the permutation vector K. Define $h_k(\underline{X}) = \underline{X}^{(1)}$, the first coordinate of $\underline{X}$, for k = 2,4,6,8 and $h_k(\underline{X}) = \underline{X}^{(2)}$, the second coordinate, for k = 1,3,5,7. Figure 1 gives a graphical representation of the rectangular coordinate method of forming blocks and Figure 2 is the representation for the polar coordinate method. The random sample that was used in both figures is found in Table I.

TABLE I: SAMPLE BIVARIATE DATA

N = 8

Observation / Coordinate	1	2	3	4	5	6	7	8
1	-3.54	2.25	-1.00	.71	2.00	- .75	-2.25	0.00
2	0.00	-2.25	0.50	.00	1.25	-1.50	-1.50	-0.50

The first element of the permutation vector is k = 2 and $h_2(\underline{X}) = \underline{X}^{(1)}$, therefore $X_2^{(1)}$ is defined to be the second smallest first coordinate. This partitions the sample space into two blocks,

$$B_1 = \left\{ X: \ X^{(1)} \leq X_2^{(1)} \right\},$$

$$B_2 = \left\{ X: \ X^{(1)} > X_2^{(1)} \right\}.$$

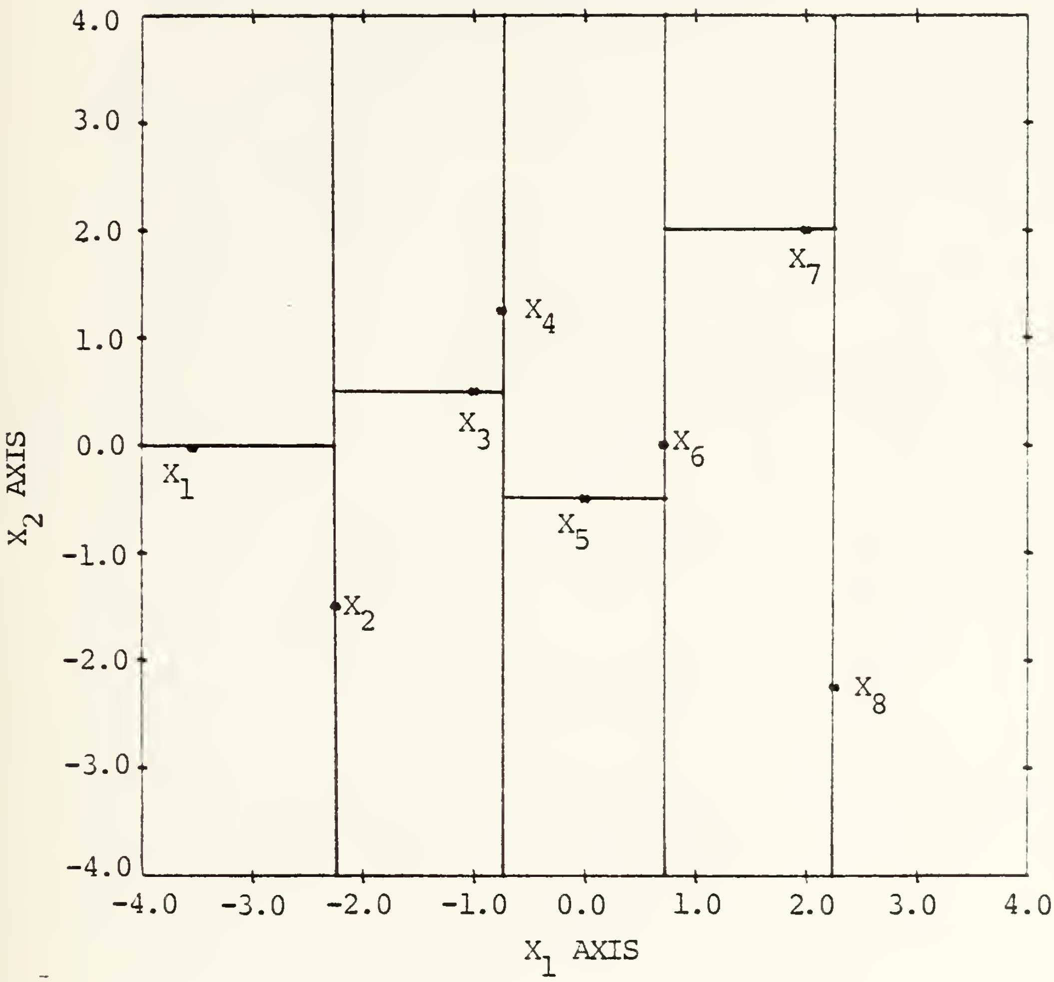


FIGURE 1: STATISTICALLY EQUIVALENT BLOCKS--
RECTANGULAR COORDINATES

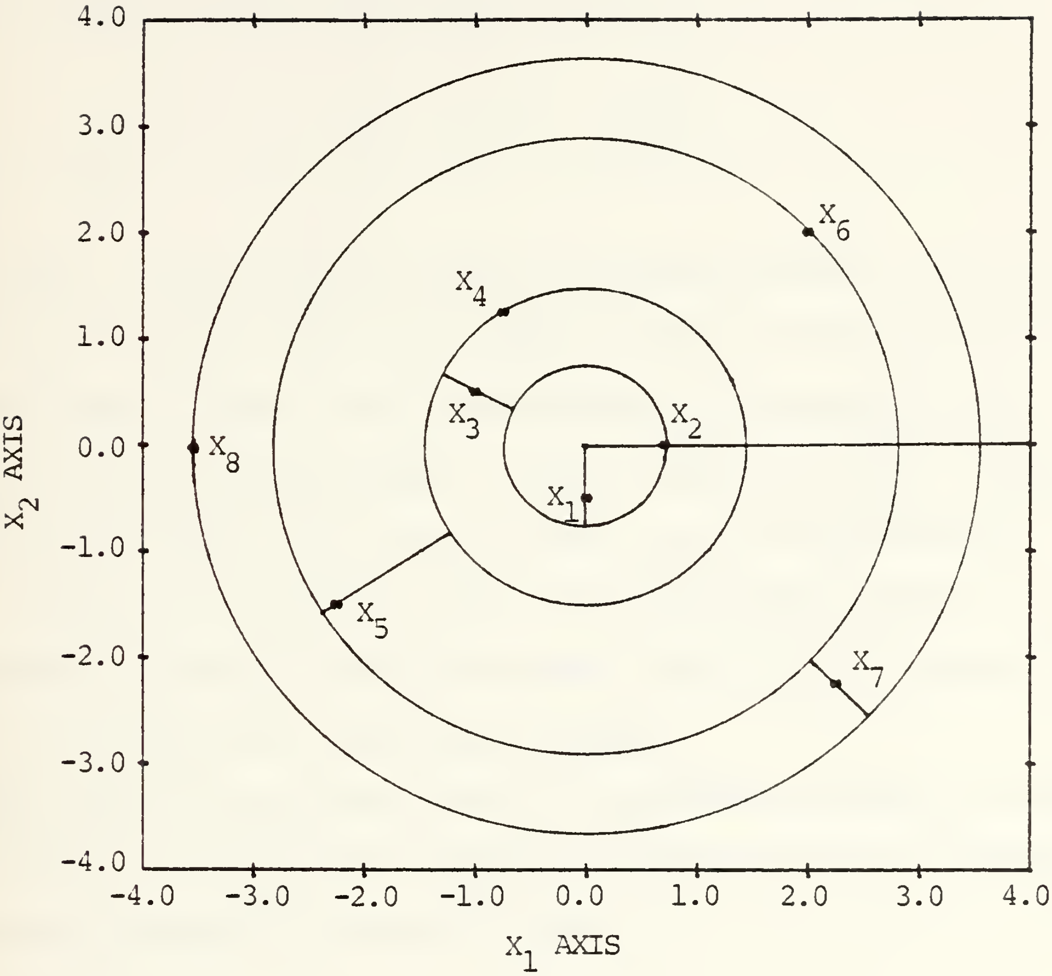


FIGURE 2: STATISTICALLY EQUIVALENT BLOCKS--
POLAR COORDINATES

The second element of the permutation vector is $k_2 = 4$, $h_4(\underline{X}) = \underline{X}^{(1)}$ and $k_2 > k_1$. Hence the block B_2 is partitioned into two sub-blocks,

$$B_{21} = B_2 \cap \left\{ X: \; X^{(1)} \leq X_2^{(1)} \right\},$$

$$B_{22} = B_2 \cap \left\{ X: \; X^{(1)} > X_2^{(1)} \right\},$$

where $X_2^{(1)}$ is the second largest coordinate among the $\underline{X}$'s in block B_2. At this stage the sample space is partitioned into three blocks. Next, the third element of the permutation vector and the corresponding cutting function define another partition of one of the three blocks into two sub-blocks. This process is continued until the permutation vector is exhausted, at which stage the sample space will be partitioned into 9 statistically equivalent blocks.

The CEDF is now constructed by spreading a mass 1/n within each block. If H_0 is the hypothesized CDF and H_n the CEDF, the test statistic Fn takes the form

$$Fn = \sup_{\underline{X}} |H_n(\underline{X}) - H_0(\underline{X})|. \qquad (1)$$

Let D_i, $i = 1,2,\ldots,n$, be the probability contents of the blocks B_i under the null hypothesis H_0, i.e., $D_i = \int_{B_i} dH_0(\underline{x})$. A computational form of the Foutz test statistic is,

$$Fn = \sum_{i=1}^{n} \text{Max}\ (0, \frac{1}{n} - D_i). \qquad (2)$$

Foutz gave the following representation for the cumulative distribution of the test statistic

$$P(Fn < x) = \int_{-\infty}^{x} \cdots \int_{-\infty}^{x} g_n(\delta_1, \delta_2, \ldots, \delta_{n-1})\, d\delta_1 d\delta_2, \ldots, d\delta_{n-1}; \qquad (3)$$

where

$$g_n(\delta_1, \delta_2, \ldots, \delta_{n-1}) = n!(n-1)!$$

for

$$\frac{1}{n} \geq \delta_1 > (\delta_2 - \delta_1) > \ldots > (\delta_{n-1} - \delta_{n-2}) > -\delta_{n-1}.$$

The evaluation of this integral is cumbersome and has not been carried out for $n > 5$. Foutz has therefore derived a large sample normal approximation given by

$$\lim_{n \to \infty} P[Fn \leq x] = \Phi\left[\frac{n^{(1/2)}(x - e^{-1})}{(2e^{-1} - 5e^{-2})^{1/2}}\right]. \qquad (4)$$

To check the accuracy of the normal approximation, Franke and Jayachandran [Ref. 4] generated 80,000 samples of sizes 20, 30 and 50. Table II contains the empirical significance

TABLE II: EMPIRICAL SIGNIFICANCE LEVEL OF THE FOUTZ Fn TEST

Sample Size Normal Significance Level	20	30	50
.10	.0757	.0800	.0859
.05	.0372	.0399	.0428
.01	.0082	.0083	.0093

levels, when the normal approximation was used to determine the critical values for the Fn test.

It is clear that the rejection rates given in Table II are consistently lower than the nominal values. More accurate critical values were therefore determined from the 80,000 Fn values and are presented in Table III.

TABLE III: APPROXIMATE CRITICAL VALUES FOR Fn TEST

Sample Size Significance Level	20	30	50
.10	.42714 (.43586)	.41903 (.42383)	.40816 (.41150)
.05	.44865 (.45513)	.43553 (.43969)	.42116 (.42386)
.01	.48659 (.49127)	.46579 (.46944)	.44487 (.44706)

Values in parentheses are those obtained from the normal approximation given by Foutz.

III. DESCRIPTION OF THE SIMULATION

In order to check the efficacy of the Foutz test as a multivariate goodness-of-fit test a simulation was run to generate sample data from various bivariate and trivariate normal distributions. The hypothesis tested in each case is that the sample is from a multivariate normal distribution with mean vector $\underline{0}$ and covariance matrix the identity $\underline{I}$. Rectangular and the polar/spherical method of blocking were both used and compared as to their effect in each case.

To validate the blocking schemes, the null hypothesis is tested against data generated from the distribution $N(\underline{0},\underline{I})$. Bivariate and trivariate sample sizes of 20, 30 and 50 are used to compute the Fn statistic which is then compared to the empirical critical levels found in Table III. Rejection rates are based on the number of rejections in 20,000 replications for each sample size. Comparing the null rejection rates to the nominal significance level used, as shown in Table III, provides evidence supporting both blocking methods as all null rejection rates are close to the significance level used.

The empirical power of the test was then investigated by varying the distribution tested. This investigation is accomplished in three different ways. First, the mean is

shifted away from the $\underline{0}$ vector while leaving the covariance as the identity matrix. This is done to investigate the ability of the test to detect location shifts. The covariance matrix is then changed from the identity while leaving the mean as the $\underline{0}$ vector. This is accomplished by changing the diagonal elements alone to investigate variance shifts and then shifting the off diagonal elements by themselves to check the effect of covariance shifts. A primary sample size of 20 was chosen for comparison and 5000 replications were used to compute rejection rates for each distribution tested. Mixing of the three types of shifts is also simulated to investigate the possible confounding effects of the three shifts. Finally sample sizes of 30 and 50 are run on a few of the distributions to determine the effect of increasing the sample size.

The various multivariate normal distributions are simulated in the following manner. Univariate normal(0,1) pseudorandom deviates are obtained from the LLRANDU series by Lewis [Ref. 5] and grouped to form a multivariate $N(\underline{0},\underline{I})$ p-variate vector. Taking the $\underline{X}^*$ so formed, the p-variate $N(\underline{0},\underline{I})$ vector random variable is transformed by

$$\underline{C}^{-1}\underline{X}^* + \underline{\mu} = \underline{X} , \qquad (1)$$

where

$$\underline{C}' \, \underline{\Sigma} \, \underline{C} = \underline{I},$$

resulting in an $\underline{X}$ which is distributed as $N(\underline{\mu},\underline{\Sigma})$. The Foutz test is then applied to each of the samples consisting of $(n-1)\ \underline{X}$s.

An example using a bivariate sample helps illustrate the blocking procedure used. Let $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}$, be the simulated bivariate sample. The first cut is made on $\underline{X}_1^{(1)}$ or the first coordinate of the first vector $\underline{X}_1$. Two blocks are formed,

		First Coordinate	Second Coordinate
B_1	=	$(-\infty, \underline{X}_1^{(1)}]$	$(-\infty, +\infty)$
B_2	=	$(\underline{X}_1^{(1)}, +\infty)$	$(-\infty, +\infty)$.

$\underline{X}_2$ is taken next and determined to be contained in block B_1 or B_2. Suppose $\underline{X}_2$ is in block B_2. B_2 is then partitioned by $\underline{X}_2^{(2)}$ or the second coordinate of sample $\underline{X}_2$. Three blocks are now defined as,

		First Coordinate	Second Coordinate
B_{10}	=	$(-\infty, \underline{X}_1^{(1)}]$	$(-\infty, +\infty)$
B_{21}	=	$(\underline{X}_1^{(1)}, +\infty)$	$(-\infty, \underline{X}_2^{(2)}]$
B_{22}	=	$(\underline{X}_1^{(1)}, +\infty)$	$(\underline{X}_2^{(2)}, +\infty)$.

This procedure is continued by examining the next vector in the random sample, locating the block that it is contained in and partitioning the block by the designated coordinate. The coordinate cutting functions used are alternated starting with the first coordinate for the first cut. Coordinate ranges, as shown, are used to designate blocks and the process is continued until n blocks are so defined. Given any random sample this method can be shown to be equivalent to a unique permutation vector K and a set of cutting functions $\{h_k\}$ as defined in Section II.

After the formation of the statistically equivalent blocks, each block has the probability content of 1/n and must be compared to the hypothesized content using the statistic

$$Fn = \sum_{i=1}^{n} \max[0, \frac{1}{n} - D_i]. \qquad (2)$$

D_i, the probability content of each block, under the null hypothesis, is defined by the integral of the null density over the block. The integral of the multivariate normal $(\underline{0}, \underline{I})$ over a rectangular block yields

$$D_i = \int \cdots \int_{B_i} (2\pi)^{\frac{-p}{2}} e^{-(1/2)\underline{x}'\underline{Ix}} d\underline{x}. \qquad (3)$$

This reduces to the product of the marginal densities which may be easily evaluated with many available routines, eliminating the need for numerical integration.

In spherical coordinates D_i is represented by

$$D_i = \int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} \int_{\rho_1}^{\rho_2} 2\pi^{(-3/2)} e^{(-1/2)\rho^2} \sin(\phi)\rho^2 \, d\rho \, d\theta \, d\phi . \tag{4}$$

Upon separation,

$$D_i = \int_{\phi_1}^{\phi_2} (1/2)\sin\phi \, d\phi \int_{\theta_1}^{\theta_2} (2\pi)^{-1} d\theta \int_{\rho_1}^{\rho_2} \frac{2\rho^2 e^{(-1/2)\rho^2} d\rho}{(2\pi)^{1/2}} . \tag{5}$$

Noting that with a change of variables the third integrand is a Chi-square density with 3 degrees of freedom, we may use a closed form expression to evaluate D as follows:

$$D_i = [\tfrac{1}{2}(\cos\phi_2 - \cos\phi_1)] \times [\tfrac{1}{2\pi}(\theta_2 - \theta_1)] \times [\chi^2_{3df}(\rho_2) - \chi^2_{3df}(\rho_1)] \tag{6}$$

where

$$\chi^2_{3df}(\rho_i) = P[\chi^2_{3df} \le \rho_i], \; i = 1,2.$$

For bivariate data the use of polar coordinates leads to similar simplification leaving D_i in the form

$$D_i = \frac{1}{2\pi}(\theta_2-\theta_1) \times [\chi^2_{2df}(R_2)] - [\chi^2_{2df}(R_1)]. \qquad (7)$$

After the calculation of the probability contents D_i for the n blocks, equation (2) is used to evaluate the Fn statistic for each generated sample. The statistic is then compared to the critical values found in Table III to decide if the null hypothesis is accepted or rejected. Rejection rates are defined by the number of rejections divided by the number of replications in a given run. The rejection rates thereby define an empirical power for the simulated distribution.

The major component of the Fortran simulation program used to evaluate the Foutz statistic for a given sample is available in the Appendix. It has been adapted for use for sample sizes up to 50, with redimensioning being needed for larger sample sizes. The program is applicable for fitting data from any hypothesized multivariate normal distribution and provides the Fn statistic as computed by both blocking methods presented. The code is self-contained except for three IMSL routines, LUDECP, MDNOR, and MDCH [Ref. 6]. These subroutines provide matrix decomposition, univariate normal probabilities and chi-square probabilities, respectively, and must be available or substituted prior to utilization of the program.

IV. RESULTS AND CONCLUSIONS

The results of the simulation are summarized in Tables IV-XIV. Rejection rates are given by the distribution tested and the significance level used. Empirical power curves are presented in Figures 3-8. Rejection rates are plotted against the magnitude of the shift in mean, variance and covariance for the distribution tested. All power curves are based on 5000 replicated samples and were compared at the α = .05 significance level.

The results for the case in which the distribution of the samples is the same as the hypothesized distribution viz., $N(\underline{0},\underline{I})$ are given in Tables IV and V. The rejection levels obtained are close to the nominal significance level for both blocking methods. No distinct pattern of variation about the prescribed levels is discernible for either method, as expected.

The rejection rates for mean shifts are given in Tables VI-VII and Figures 3-4. Shifts in the mean vector are detected well; a shift of one standard deviation in a single coordinate resulted in a 60% rejection rate for bivariate or trivariate data. Greater shifts in mean led to even higher rejection rates. The rectangular method of blocking consistently gave about a 10% improvement over the polar/spherical method in detecting mean shifts.

Results for variance shifts are contained in Tables VIII and IX and the power curves are given in Figures 5 and 6. The Foutz test did not detect small variance shifts very well but the performance of the test was far better for larger shifts or shifts in more than one coordinate. No one method of blocking performed better in all cases but in general the polar/spherical method seemed to outperform the rectangular method for detecting variance shifts.

The results for changes in covariance are summarized in Tables X, XI and Figure 7. Covariance shifts are not detected well for either blocking method except for highly correlated data with the correlation coefficient equal to .9. The polar/spherical coordinate blocking method appeared to perform a little better than the rectangular coordinate method of blocking, but in general the simulation revealed that the Fn test is not very powerful against covariance shifts.

The empirical power for combinations of shifts in mean and variance or covariance are presented in Tables XII and XIII. Entries are based on an $\alpha = .05$ significance level and are tabled by the mean vector and covariance matrix of the sample data. Entries farther down and to the right correspond to greater shifts in mean and variance/covariance and are generally larger, as is to be expected. There are no apparent confounding problems due to shifts in both

parameters. The rectangular method of blocking, however, did outperform the polar/spherical method for most cases of multiple shifts.

The results indicative of the effect of increasing the sample size are summarized in Tables XIV and XV. Results for sample sizes of 20, 30, and 50 are given for some representative cases. The tables reveal higher rejection rates for larger sample sizes with increases being comparable for both blocking methods.

This study was limited to the two and three variate normal distribution. There are many problems for further research. Of primary concern is the generation of percentage points of Fn for various values of n. The intractability of the problem of obtaining the exact distribution requires an empirical approach to finding a correction to the asymptotic approximation given by Foutz. Since the use of coordinates as cutting functions worked well, the method should be tried for other distributions and higher dimensions.

In conclusion, the Fn test is found to be a viable option for testing goodness-of-fit of multivariate normal distributions. These encouraging empirical results indicate further study should be conducted to explore the potential of this test for other distributions.

TABLE IV: NULL EMPIRICAL REJECTION LEVELS FOR THE BIVARIATE NORMAL DISTRIBUTION

Significance Level Blocking Method	.01	.05	.10
		N = 20	
Rectangular	.0098	.0488	.0940
Polar	.0096	.0482	.1020
		N = 30	
Rectangular	.0110	.0510	.0944
Polar	.0082	.0454	.0890
		N = 50	
Rectangular	.0120	.0498	.0950
Polar	.0098	.0484	.0958

BASED ON 20,000 REPLICATIONS

TABLE V: NULL EMPIRICAL REJECTION LEVELS FOR THE TRIVARIATE NORMAL DISTRIBUTION

Significance Level Blocking Method	.01	.05	.10
		N = 20	
Rectangular	.0104	.0440	.0982
Spherical	.0120	.0518	.1048
		N = 30	
Rectangular	.0114	.0480	.0956
Spherical	.0140	.0484	.0914
		N = 50	
Rectangular	.0098	.0484	.0960
Spherical	.0088	.0478	.0914

BASED ON 20,000 REPLICATIONS

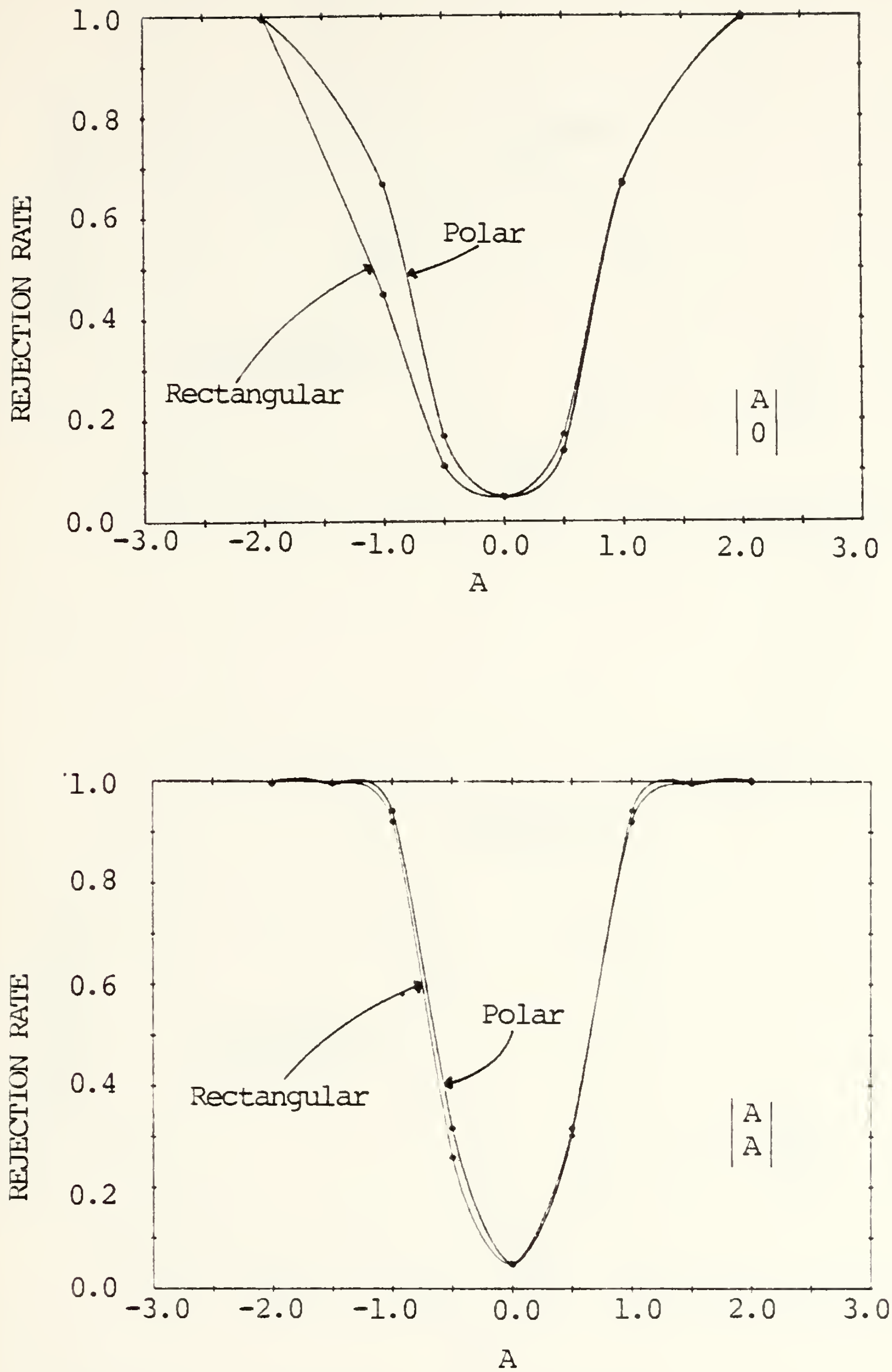
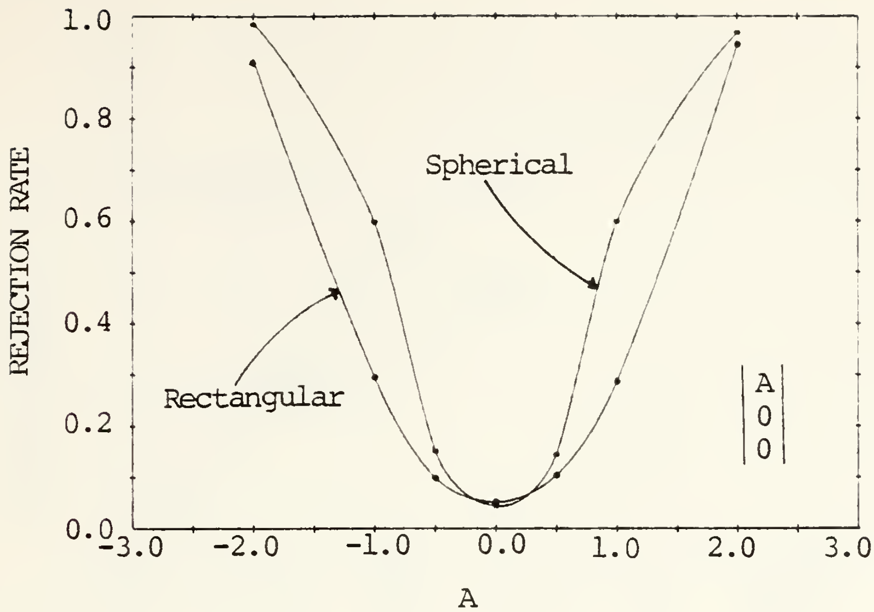


FIGURE 3: POWER CURVES FOR SHIFTS IN MEAN (BIVARIATE)



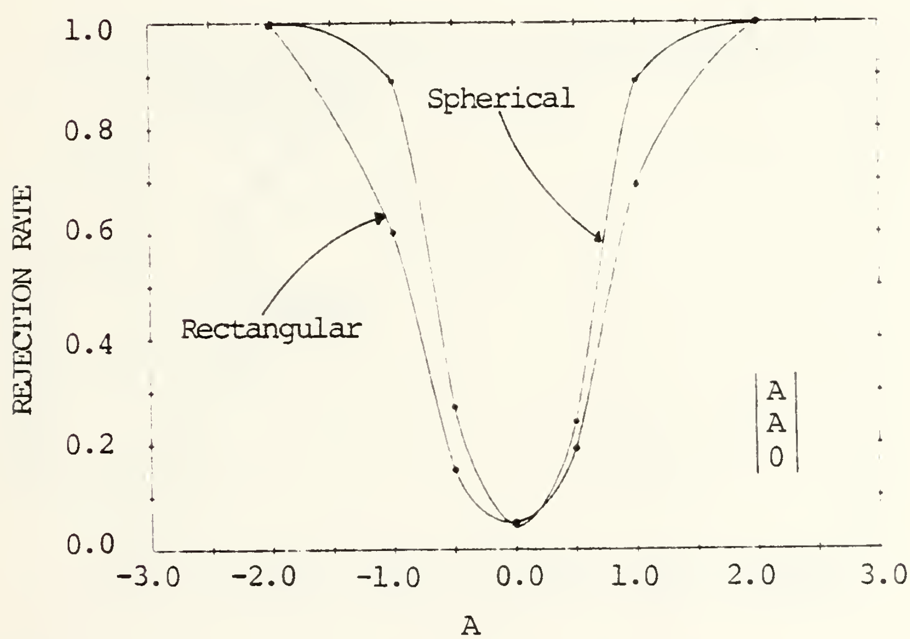
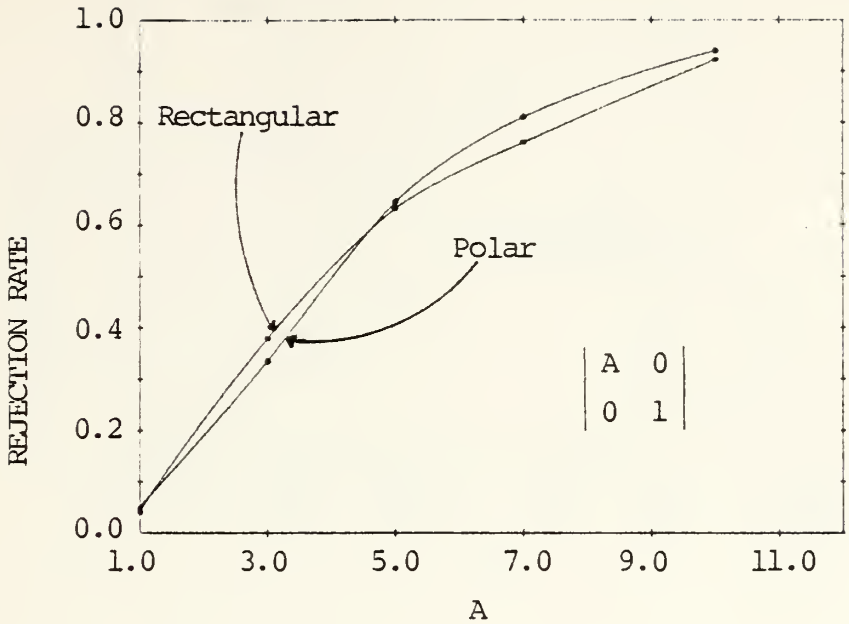


FIGURE 4: POWER CURVES FOR SHIFTS IN MEAN (TRIVARIATE)



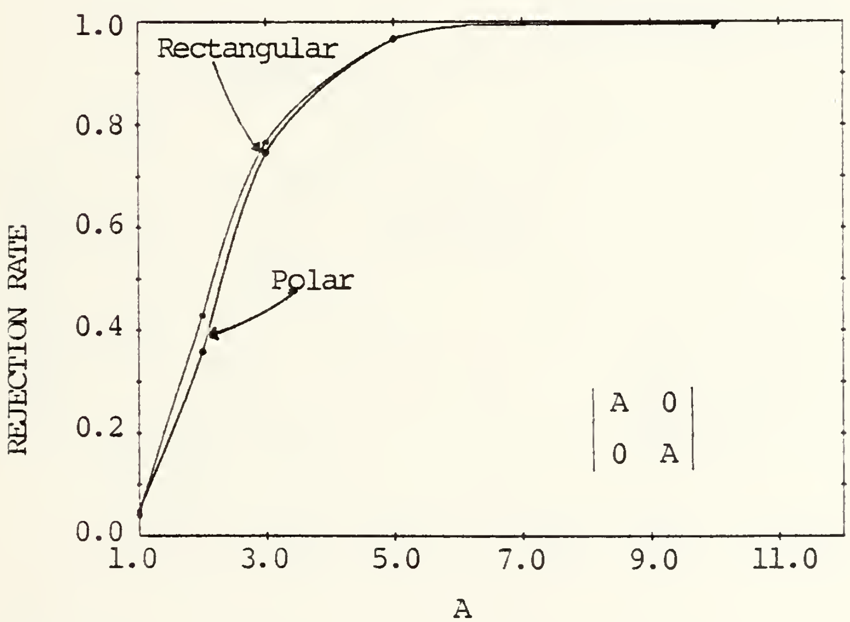
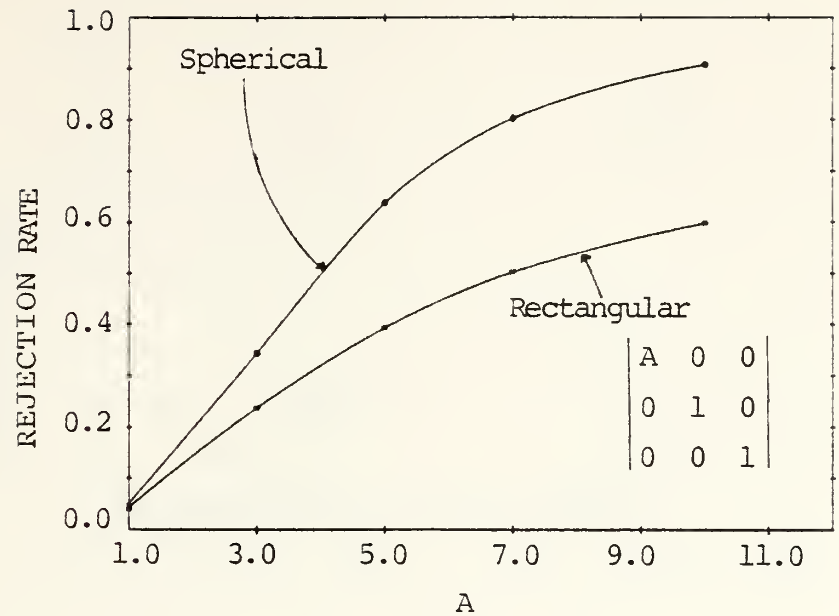


FIGURE 5: POWER CURVES FOR SHIFTS IN VARIANCE (BIVARIATE)



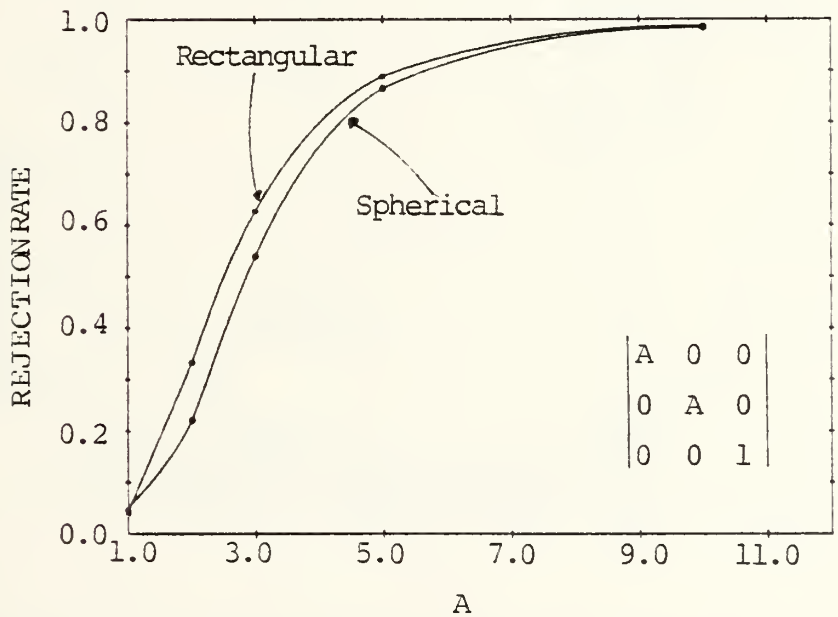
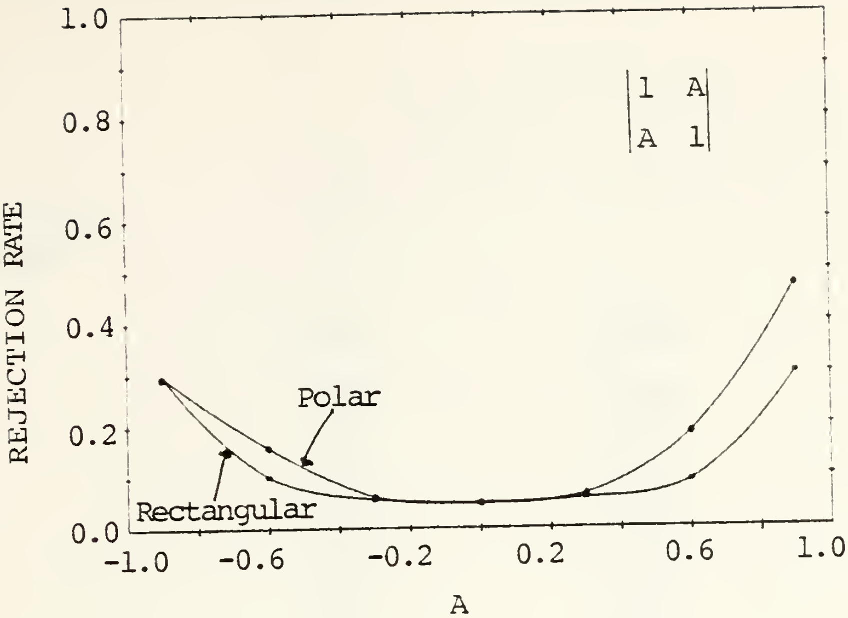


FIGURE 6: POWER CURVES FOR SHIFTS IN VARIANCE (TRIVARIATE)



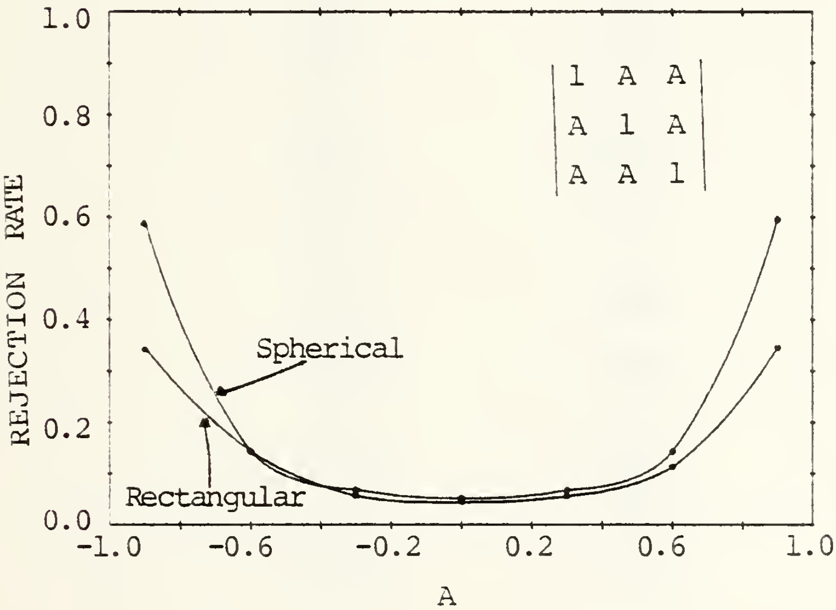


FIGURE 7: POWER CURVES FOR SHIFTS IN COVARIANCE

TABLE VI: REJECTION RATES FOR SHIFTS IN MEAN (BIVARIATE)

N = 20

Critical Value Mean Tested	.01	.05	.10
0 0	.0094 .0102	.0488 .0482	.0998 .0996
-.5 0	.0566 .0346	.1684 .1096	.2816 .2138
.5 0	.0574 .0430	.1710 .1388	.2700 .2298
-.5 -.5	.1408 .1038	.3164 .2592	.4534 .3610
.5 .5	.1294 .1230	.3024 .3164	.4406 .4534
-1 0	.4357 .2340	.6664 .4484	.7834 .6046
1 0	.4464 .2444	.6700 .6664	.7842 .7834
-1 -1	.8382 .7780	.9418 .9212	.9748 .9610
1 1	.8428 .6930	.9418 .9212	.9718 .9610
-2 0	.9936 .9926	.9980 .9990	.9996 .9996
2 0	.9948 .9762	.9998 .9950	1.0000 .9980
-2 -2	1.0000 1.0000	1.0000 1.0000	1.0000 1.0000
2 2	1.0000 1.0000	1.0000 1.0000	1.0000 1.0000

BASED ON 5000 REPLICATIONS

FIRST ENTRY--RECTANGULAR
SECOND ENTRY--POLAR

TABLE VII: REJECTION RATES FOR SHIFTS IN MEAN (TRIVARIATE)

N = 20

Critical Value Mean Tested	.01	.05	.10
0 0 0	.0104 .0120	.0440 .0518	.0982 .1048
-.5 0 0	.0492 .0216	.1502 .0980	.2474 .1632
.5 0 0	.0480 .0280	.1438 .1036	.2484 .1874
-.5 -.5 0	.1076 .0472	.2704 .1516	.3990 .2516
.5 .5 0	.0972 .0658	.2424 .1912	.3688 .3046
-.5 -.5 -.5	.1826 .0848	.3788 .2198	.5170 .3584
.5 .5 .5	.1738 .0848	.3642 .2198	.4948 .3584
-1 0 0	.3782 .1184	.5984 .2942	.7212 .4212
1 0 0	.3728 .1174	.5984 .2866	.7212 .4234
-1 -1 0	.7392 .3808	.8892 .6020	.9410 .7338
1 1 0	.7400 .4670	.8892 .6918	.9410 .7934

TABLE VII (Continued)

Critical Value	.01	.05	.10
Mean Tested			
-2	.9636	.9872	.9958
0	.7772	.9138	.9916
0			
2	.8992	.9676	.9832
0	.8486	.9448	.9736
0			
-1	.9134	.9778	.9894
-1	.7688	.8744	.9312
-1			
1	.9102	.9746	.9900
1	.7936	.9244	.9598
1			
-2	1.0000	1.0000	1.0000
-2	.9984	.9996	1.0000
0			
2	1.0000	1.0000	1.0000
2	.9998	1.0000	1.0000
0			
-2	1.0000	1.0000	1.0000
-2	1.0000	1.0000	1.0000
-2			
2	1.0000	1.0000	1.0000
2	1.0000	1.0000	1.0000
2			

BASED ON 5000 REPLICATIONS

FIRST ENTRY--RECTANGULAR
SECOND ENTRY--SPHERICAL

TABLE VIII. REJECTION RATES FOR SHIFTS IN VARIANCE (BIVARIATE)

N = 20

Critical Values		.01	.05	.10
Variance Tested				
1	0	.0094	.0488	.0998
0	1	.0102	.0482	.0996
1	0	.1864	.3786	.5150
0	3	.1578	.3342	.4640
2	0	.2228	.4292	.5628
0	2	.1714	.3582	.4928
1	0	.4030	.6322	.7474
0	5	.4286	.6448	.7574
3	0	.5790	.7666	.8580
0	3	.5338	.7450	.8368
1	0	.5640	.7608	.8556
0	7	.6312	.8106	.8856
1	0	.7092	.8618	.9222
0	10	.8088	.9228	.9600
5	0	.8998	.9664	.9832
0	5	.8998	.9665	.9804
10	0	.9956	.9994	.9998
0	10	.9920	.9978	.9988

BASED ON 5000 REPLICATIONS

FIRST ENTRY--RECTANGULAR
SECOND ENTRY--POLAR

TABLE IX. REJECTION RATES FOR SHIFTS IN VARIANCE (TRIVARIATE)

N = 20

Critical Value			.01	.05	.10
Variance Tested					
1	0	0	.0104	.0440	.0982
0	1	0	.0120	.0518	.1048
0	0	1			
3	0	0	.0924	.2372	.3626
0	1	0	.1606	.3438	.4736
0	0	1			
2	0	0	.1500	.3330	.4644
0	2	0	.0888	.2208	.3308
0	0	1			
5	0	0	.1940	.3832	.5372
0	1	0	.4146	.6374	.7550
0	0	1			
7	0	0	.2708	.5026	.6332
0	1	0	.6100	.8032	.8792
0	0	1			
2	0	0	.2758	.5012	.6326
0	2	0	.2510	.4538	.5814
0	0	2			
3	0	0	.4140	.6270	.7514
0	3	0	.3240	.5394	.6566
0	0	1			
3	0	0	.6622	.8372	.9038
0	3	0	.6752	.8312	.8966
0	0	3			
10	0	0	.3716	.5980	.7186
0	1	0	.7880	.9078	.9506
0	0	1			
5	0	0	.7558	.8896	.9346
0	5	0	.7256	.8660	.9182
0	0	1			
5	0	0	.9390	.9770	.9872
0	5	0	.9292	.9762	.9852
0	0	5			

TABLE IX (Continued)

N = 20

Critical Value Variance Tested	.01	.05	.10
10 0 0 0 10 0 0 0 0	.9572 .9470	.9866 .9832	.9950 .9926
10 0 0 0 10 0 0 0 10	.9972 .9858	.9998 .9970	1.0000 .9994

BASED ON 5000 REPLICATIONS

FIRST ENTRY--RECTANGULAR
SECOND ENTRY--SPHERICAL

TABLE X: REJECTION RATES FOR SHIFTS IN COVARIANCE (BIVARIATE)

N = 20

Critical Value Covariance Tested		.01	.05	.10
1	0	.0094	.0488	.0998
0	1	.0102	.0482	.0996
1	-.3	.0152	.0558	.1068
-.3	1	.0126	.0598	.1274
1	.3	.0126	.0576	.1178
.3	1	.0136	.0656	.1258
1	-.6	.0288	.1008	.1782
-.6	1	.0514	.1576	.2560
1	.6	.0250	.0912	.1702
.6	1	.0648	.1838	.2984
1	-.9	.1166	.2996	.4446
-.9	1	.2378	.2982	.6162
1	.9	.1122	.2996	.4446
.9	1	.2378	.4710	.6042

BASED ON 5000 REPLICATIONS

FIRST ENTRY--RECTANGULAR
SECOND ENTRY--POLAR

TABLE XI: REJECTION RATES FOR SHIFTS IN COVARIANCE (TRIVARIATE)

N = 20

Critical Value			.01	.05	.10
Covariance Tested					
1	0	0	.0104	.0440	.0982
0	1	0	.0120	.0518	.1048
0	0	1			
1	0	-.3	.0104	.0540	.1076
0	1	0	.0124	.0488	.1066
-.3		1			
1	0	.3	.0106	.0468	.0972
0	1	0	.0124	.0512	.1086
.3	0	1			
1	.3	.3	.0126	.0560	.1112
.3	1	.3	.0162	.0676	.1292
.3	.3	1			
1	0	-.6	.0152	.0740	.1394
0	1	0	.0122	.0584	.1158
-.6	0	1			
1	0	.6	.0148	.0674	.1298
0	1	0	.0128	.0582	.1194
.6	0	1			
1	.6	.6	.0308	.1136	.1960
.6	1	.6	.0432	.1434	.2486
.6	.6	1			
1	0	-.9	.0412	.1358	.2314
0	1	0	.0254	.0974	.1842
-.9	0	1			
1	0	.9	.0402	.1386	.2368
0	1	0	.0258	.1386	.2368
.9	0	1			
1	.9	.9	.1406	.3454	.4942
.9	1	.9	.3646	.5950	.7122
.9	.9	1			

BASED ON 5000 REPLICATIONS

FIRST ENTRY--RECTANGULAR
SECOND ENTRY--SPHERICAL

TABLE XII: REJECTION RATES FOR MULTIPLE SHIFTS IN MEAN AND VARIANCE-COVARIANCE (BIVARIATE)

N = 20

Sigma	1 0 0 1	1 .6 .6 1	2 0 0 2	2 .849 .849 2	5 1.34 1.34 5
Mean					
0 0	.0488 .0482	.0912 .1176	.1986 .1522	.2500 .2162	.9658 .9572
.5 0	.1710 .1388	.2398 .2482	.3110 .2498	.3702 .3402	.9720 .9650
1 0	.5606 .4348	.7346 .5952	.6384 .5334	.6828 .6316	.9820 .9764
1 1	.9418 .8576	.8774 .8588	.9350 .8722	.8658 .8308	.9892 .9840
2 0	.9998 .9950	.9998 .9990	.9902 .9772	.9950 .9882	.9990 .9964

FIRST ENTRY--RECTANGULAR
SECOND ENTRY--POLAR

BASED ON 5000 REPLICATIONS

$\alpha = .05$

TABLE XIII: REJECTION RATES FOR MULTIPLE SHIFTS IN MEAN AND VARIANCE-COVARIANCE (TRIVARIATE)

N = 20

Sigma	1 0 0 0 1 0 0 0 1	1 0 .6 0 1 0 .6 0 1	5 0 0 0 1 0 0 0 1	10 0 .95 0 1 0 .95 0 1	5 0 0 0 5 0 0 0 5
Mean					
0 0 0	.0440 .0518	.0674 .0582	.5392 .4584	.7828 .7840	.9770 .9720
.5 0 0	.0480 .0280	.1830 .1176	.5708 .5034	.7946 .8020	.9832 .9740
1 0 0	.3728 .1174	.6352 .2912	.6852 .6254	.8206 .8422	.9888 .9824
1 1 0	.7400 .7392	.9074 .7454	.9270 .8602	.9668 .9454	.9930 .9870
2 0 1	.9982 .9726	.9956 .9752	.9716 .9742	.9736 .9774	.9978 .9976

FIRST ENTRY--RECTANGULAR
SECOND ENTRY--SPHERICAL

BASED ON 5000 REPLICATIONS

$\alpha = .05$

TABLE XIV: REJECTION RATES FOR INCREASING SAMPLE SIZES (BIVARIATE)

Sample size Shift	20	30	50
		$\alpha = .01$	
.5	.0574	.0860	.1270
0	.0430	.0564	.0754
.5	.1294	.2026	.3652
.5	.1230	.1418	.2508
1 .3	.0126	.0140	.0176
.3 1	.0136	.0152	.0170
1 0	.1864	.2722	.4522
0 3	.1578	.2244	.3744
		$\alpha = .05$	
.5	.1710	.2170	.2914
0	.1388	.1630	.2238
.5	.3024	.4144	.6030
.5	.3164	.3076	.4826
1 .3	.0576	.0624	.0728
.3 1	.0656	.0624	.0760
1 0	.3786	.4884	.6756
0 3	.3342	.4304	.6016
		$\alpha = .10$	
.5	.2700	.3228	.4256
0	.2298	.2658	.3400
.5	.4406	.5424	.7190
.5	.4534	.4336	.6066
1 .3	.1178	.1174	.1396
.3 1	.1258	.1196	.1422
1 0	.5150	.6132	.7800
0 3	.4640	.5566	.7160

BASED ON 5000 REPLICATIONS

FIRST ENTRY--RECTANGULAR
SECOND ENTRY--POLAR

TABLE XV: REJECTION RATES FOR INCREASING SAMPLE SIZES (TRIVARIATE)

Sample size / Shift	20	30	50
		$\alpha = .01$	
.5 0 0	.0480 .0280	.0680 .0362	.1036 .0526
.5 .5 .5	.1738 .0848	.2932 .1662	.5040 .3428
1 0 .3 0 1 0 .3 0 1	.0106 .0124	.0138 .0134	.0148 .0144
3 0 0 0 1 0 0 0 1	.1606 .0838	.2054 .1138	.3528 .2080
		$\alpha = .05$	
.5 0 0	.1438 .1036	.1974 .1256	.2742 .1646
.5 .5 .5	.3642 .2198	.5118 .3588	.7268 .5868
1 0 .3 0 1 0 .3 0 1	.0468 .0512	.0588 .0488	.0656 .0540
3 0 0 0 1 0 0 0 1	.3438 .2074	.3976 .2708	.5734 .4126
		$\alpha = .10$	
.5 0 0	.2484 .1874	.3024 .2132	.3876 .2650
.5 .5 .5	.4948 .3584	.6396 .4912	.8272 .7040
1 0 .3 0 1 0 .3 0 1	.0972 .1086	.1142 .1030	.1232 .1102
3 0 0 0 1 0 0 0 1	.4736 .3156	.5264 .3880	.6880 .5450

BASED ON 5000 REPLICATIONS FIRST ENTRY: RECTANGULAR
SECOND ENTRY: SPHERICAL

APPENDIX A

USER REQUIREMENTS AND INPUT FORMAT FOR PROGRAM FOUTZ

The use of the Computer program contained in Appendix B requires the sample size, number of variates, applicable data and the Multivariate Normal distribution being tested as described by the mean vector and the variance-covariance matrix. The variables containing the required inputs as well as the required input format are as shown below.

DESCRIPTION OF VARIABLES

```
N------------------Sample size
M------------------Number of Variables (2 or 3)
SIGMA1-------------Variance-Covariance Matrix
                   (MxM)
B1-----------------Mean Vector (Mx1)
X------------------Matrix of Sample Data (MxN)
```

INPUT FORMAT

```
N,M----------------(2I5)
SIGMA1-------------(3F12.6)   Input M Rows
B1-----------------(F12.6)    Input M Rows
X------------------(3F12.6)   Input M Rows
```

Input data is echoed in the output providing a check for correct entry of data as well as is the decomposition of SIGMA1. The Fn statistic as computed by both methods of blocking follows completing the output given for a single run. An example run is given for Trivariate data of sample size 10.

SAMPLE TRIVARIATE RUN

```
FOUTZ TEST FOR  3 VARIATE NORMAL
THE NUMBER OF OBSERVATIONS = 10
OBSERVATIONS ENTERED AS FOLLOWS
    3.170000     7.540000     4.230000
    4.160000     5.500000     5.580000
    2.330000     2.910000     6.620000
    2.530000     3.440000     5.660000
    1.990000     2.630000     6.320000
    2.260000     2.800000     6.730000
    2.630000     0.290000     6.550000
    3.440000     4.860000     3.150000
    3.500000     4.670000     8.310000
    3.580000     3.230000     4.970000
DISTRIBUTION TESTED
COVARIANCE MATRIX
    1.000000     1.000000     1.000000
    1.000000     3.000000     1.000000
    1.000000     1.000000     5.000000
MEAN VECTOR
    2.000000
    3.000000
    4.000000
DECOMPOSITION OF SIGMA
    1.000000     0.0          0.0
   -0.707107     0.707107     0.0
   -0.500000     0.0          0.500000

WITH POLAR OR SPHERICAL COORDINATES
FOUTZ STAT=      0.593289
WITH RECTANGULAR COORDINATES
FOUTZ STAT=      0.556877
```

APPENDIX B

CCMPUTER PRCGRAM FOUTZ

```
C                 ******************************
C                 *         FORTRAN CODE       *
C                 *            FCUTZ           *
C                 ******************************
C
C
C
C
C       N                    SAMPLE SIZE
C       M                    DIMENSICN CF EACH VECTOR
C       IRAD                 N VECTOR DESIGNATING COORDINATE TO CUT ON
C       SIGMA1               (M,M)COVARIANCE MATRIX  TEST DISTRIBUTION
C       B1                   (M,1)MEAN VECTOR
C
C
C                                MAIN PRCGRAM
C
C       PURPOSE:
C               READS IN N,M AND DIMENSIONS
C               LIMITED TO N=50,M=3 AS SET
C
        DIMENSICN IRAD(52),VECT(50,6),WKVEC(6),BLCCK(51,12),
       $SIGMA1(3,3),B1(3,1),X(50,3),TRAN(3,1),XTT(3,1),C(3,3),
       $BLCC(51,12),XTTR(3,1)
        READ(5,990)N,M
990     FORMAT(2I5)
        NN=N+1
        MM=2*M
        NM1=N-1
        DO 10 I=1,N,M
        DO 5 J=1,M
        IRAD(J+I-1)=J
5       CCNTINUE
10      CONTINUE
        CALL DDRIVE(IRAD,VECT,WKVEC,BLOCK,BLOC,SIGMA1,B1,N,M,
       $NN,MM,TRAN,XTT,C,XTTR,X)
        STOP
        END
C.......................................................................
C               SUBRCUTINE DCRIVE
C
C               PURPOSE:
C               DRIVES PRCGRAM AND VARIABLE DIMENSINS BASED ON
C               M ANC N.  READS IN B1, SIGMA1 AND DATA TO BE
C               TESTED.  ECHCS INPUT DATA AND PRINTS THE
C               RESULTING FN STATISTIC.
C.......................................................................
C
        SUBROUTINE DDRIVE(IRAD,VECT,WKVEC,BLCCK,BLOC,SIGMA1,
       $B1,N,M,NN,MM,TRAN,XTT,C,XTTR,X)
        DIMENSICN IRAC(N),VECT(N,M),WKVEC(3),BLOCK(NN,6),
       $SIGMA1(M,M),B1(M,1),TRAN(M,1),X(N,M),BLOC(NN,MM),
       $XTT(M,1),C(M,M),XTTR(M,1)
        DO 30 I=1,M
        READ(5,992)(SIGMA1(I,J),J=1,M)
30      CONTINUE
992     FORMAT(3F12.6)
        DO 40 I=1,M
```

```
      READ(5,993)B1(I,1)
993   FORMAT(F12.6)
40    CONTINUE
      DO 70 I=1,N
      READ(5,992)(X(I,J),J=1,M)
70    CONTINUE
CC    ECHO INPUT DATA
      WRITE(6,800)M
      WRITE(6,801)N
800   FORMAT('1','FCUTZ TEST FOR ',I2,' VARIATE NORMAL')
801   FORMAT('0','THE NUMBER OF OBSERVATIONS =',I3)
804   FORMAT(' ',3F12.6)
805   FORMAT('0','MEAN VECTOR')
806   FORMAT(' ',F12.6)
      WRITE(6,791)
791   FORMAT('0','OBSERVATIONS ENTERED AS FOLLOWS')
      DO 792 I=1,N
792   WRITE(6,804)(X(I,J),J=1,M)
      WRITE(6,807)
      WRITE(6,808)
807   FORMAT('0','DISTRIBUTION TESTED')
808   FORMAT('0','COVARIANCE MATRIX')
      DO 793 I=1,M
793   WRITE(6,804)(SIGMA1(I,J),J=1,M)
      WRITE(6,805)
      WRITE(6,806)((B1(I,J),J=1,1),I=1,M)
      CALL DECOMP(SIGMA1,M,C,0)
C     CALL TRANSFORMATION  ROUTINES
      DO 750 I=1,N
      DO 751 J=1,M
 751  XTT(J,1)=X(I,J)
 752  FORMAT(' ',3X,F12.6)
      CALL TRANS(M,XTT,B1,TRAN,C,XTTR)
      DO 760 J=1,M
      VECT(I,J)=TRAN(J,1)
 760   CONTINUE
 750   CONTINUE
      CALL BLOCKS(VECT,N,NN,M,MM,IRAD,BLOCK,WKVEC)
      CALL BLOCKR(VECT,N,NN,M,MM,IRAD,BLOC,WKVEC)
C
C BLOCK BY POLAR OR SPHERICAL COORDINATES
      CALL FOUTZ(BLOCK,NN,MM,FN,M)
      WRITE(6,989)
989   FORMAT('0','WITH POLAR OR SPHERICAL COORDINATES')
      WRITE(6,990)FN
990   FORMAT(' ','FCUTZ STAT= ',F12.6)
C BLOCK BY RECTANGULAR COORDINATES
      CALL FOUTR(BLOC,NN,MM,FN,M)
      WRITE(6,988)
988   FORMAT('0','WITH RECTANGULAR COORDINATES')
      WRITE(6,991)FN
991   FORMAT(' ','FCUTZ STAT= ',F12.6)
      RETURN
      END
C.................................................................
C                SUBROUTINE DECOMP
C
C                PURPOSE:
C                DECOMPOSES THE COVARIANCE MATRIX ENTERED.
C                USES CHOLESKY DECOMOSITION VIA IMSL ROUTINE
C                LUDECP TO PROVIDE A MATRIX C NEEDED BY THE
C                ROUTINE TRANS.
C.................................................................
C
      SUBROUTINE DECOMP(SIGMA,M,C,IV)
      DIMENSION SIGMA(M,M),C(M,M),A(51),UL(51),L1(6),M1(6)
      IJ=1
```

```
      DO 100 I=1,M
      DO 110 J=1,I
      A(IJ)=SIGMA(I,J)
      IJ=IJ+1
110   CONTINUE
100   CONTINUE
      CALL LUDECP(A,UL,M,D1,D2,IER)
      DO 120 I=1,M
      DO 130 J=1,M
      II=I*(I-1)/2+J
      IF(J.LT.I)C(I,J)=UL(II)
      IF(J.EQ.I)C(I,J)=1./UL(II)
      IF(J.GT.I)C(I,J)=0.0
130   CONTINUE
120   CONTINUE
      IF(IV.EQ.1)GO TO 121
      CALL INVT(C,M,D,L1,M1)
      WRITE(6,799)
799   FORMAT('0','DECOMPCSITICN OF SIGMA')
      DO 765 I=1,M
765   WRITE(6,700)(C(I,J),J=1,M)
700   FORMAT(' ',3F12.6)
121   CONTINUE
      RETURN
      END
C
C     .................................................
C
C        SUBRCUTINE INVT
C
C        PURPCSE
C           INVERT A MATRIX
C
C     ................................................
C
      SUBROUTINE INVT(A,N,D,L,M)
      DIMENSICN A(1),L(1),M(1)
C
C        SEARCH FOR LARGEST ELEMENT
C
      D=1.0
      NK=-N
      DO 80 K=1,N
      NK=NK+N
      L(K)=K
      M(K)=K
      KK=NK+K
      BIGA=A(KK)
      DO 20 J=K,N
      IZ=N*(J-1)
      DO 20 I=K,N
      IJ=IZ+I
   10 IF( ABS(BIGA)- ABS(A(IJ))) 15,20,20
   15 BIGA=A(IJ)
      L(K)=I
      M(K)=J
   20 CONTINUE
C
C        INTERCHANGE ROWS
C
      J=L(K)
      IF(J-K) 35,35,25
   25 KI=K-N
      DO 30 I=1,N
      KI=KI+N
      HOLD=-A(KI)
      JI=KI-K+J
```

```
      A(KI)=A(JI)
   30 A(JI) =HCLD
C
C        INTERCHANGE COLUMNS
C
   35 I=M(K)
      IF(I-K) 45,45,38
   38 JP=N*(I-1)
      DO 40 J=1,N
      JK=NK+J
      JI=JP+J
      HOLD=-A(JK)
      A(JK)=A(JI)
   40 A(JI) =HCLD
C
C        DIVIDE COLUMN BY MINUS PIVOT (VALUE OF PIVOT
C        ELEMENT IS CONTAINED IN BIGA)
C
   45 IF(BIGA) 48,46,48
   46 D=0.0
      RETURN
   48 DO 55 I=1,N
      IF(I-K) 50,55,50
   50 IK=NK+I
      A(IK)=A(IK)/(-BIGA)
   55 CONTINUE
C
C        REDUCE MATRIX
C
      DO 65 I=1,N
      IK=NK+I
      HOLD=A(IK)
      IJ=I-N
      DO 65 J=1,N
      IJ=IJ+N
      IF(I-K) 60,65,60
   60 IF(J-K) 62,65,62
   62 KJ=IJ-I+K
      A(IJ)=HCLD*A(KJ)+A(IJ)
   65 CONTINUE
C
C        DIVIDE ROW BY PIVOT
C
      KJ=K-N
      DO 75 J=1,N
      KJ=KJ+N
      IF(J-K) 70,75,70
   70 A(KJ)=A(KJ)/BIGA
   75 CONTINUE
C
C        PRODUCT OF PIVCTS
C
      D=D*BIGA
C
C        REPLACE PIVOT BY RECIPROCAL
C
      A(KK)=1.0/BIGA
   80 CONTINUE
C
C        FINAL ROW AND COLUMN INTERCHANGE
C
      K=N
  100 K=(K-1)
      IF(K) 150,150,105
  105 I=L(K)
      IF(I-K) 120,120,108
  108 JQ=N*(K-1)
```

```
      JR=N*(I-1)
      DO 110 J=1,N
      JK=JQ+J
      HOLD=A(JK)
      JI=JR+J
      A(JK)=-A(JI)
  110 A(JI) =HCLD
  120 J=M(K)
      IF(J-K) 100,100,125
  125 KI=K-N
      DO 130 I=1,N
      KI=KI+N
      HOLD=A(KI)
      JI=KI-K+J
      A(KI)=-A(JI)
  130 A(JI) =HCLD
      GO TO 100
  150 RETURN
      END
C.......................................................................
C               SUBROUTINE TRANS
C
C     PURPOSE: TC TRANSFCRM OBSERVATIONS TC N(O,I)
C              UNDER THE NULL HYPOTHESIS.  USES INPUT
C              VALUES OF B1 ANC THE MATRIX C FROM DECOMP
C              TO TRANSFCRM THE DATA ENTEREC USING,
C                     *
C                    X  = C(X-B1).
C.......................................................................
C
      SUBROUTINE TRANS(M,XTT,B1,TRAN,C,XTTR)
      DIMENSICN B1(M,1),XTT(M,1),XTTR(M,1),TRAN(M,1),C(M,M)
      CALL SUB(XTT,B1,XTTR,M,1)
      CALL PRD(C,XTTR,TRAN,M,M,1)
      RETURN
      END
C     ..................................................................
C
C        SUBRCUTINE SUB
C
C        PURPCSE
C           SUBTRACT ONE MATRIX FROM ANOTHER TO
C           FCRM RESULTANT MATRIX.
C     ..................................................................
C
      SUBROUTINE SUB(A,B,R,N,M)
      DIMENSION A(1),B(1),R(1)
C
C        CALCULATE NUMBER CF ELEMENTS
C
      NM=N*M
C
C        SUBTRACT MATRICES
C
      DO 10 I=1,NM
   10 R(I)=A(I)-B(I)
      RETURN
      END
C
C     ...................................................................
C
C        SUBRCUTINE PRD
C
C        PURPCSE
C           MULTIPLY TWO MATRICES TO FORM A
C           RESULTANT MATRIX.
C
```

```
C     ...............................................................
C
      SUBROUTINE PRC(A,B,R,N,M,L)
      DIMENSION A(1),B(1),R(1)
C
      IR=0
      IK=-M
      DO 10 K=1,L
      IK=IK+M
      DO 10 J=1,N
      IR=IR+1
      JI=J-N
      IB=IK
      R(IR)=0
      DO 10 I=1,M
      JI=JI+N
      IB=IB+1
   10 R(IR)=R(IR)+A(JI)*B(IB)
      RETURN
      END
C.................................................................
C                SUBROUTINE BLOCKS
C
C   PURPOSE:
C   THIS SUBROUTINE TAKES N M-VARIATE VECTORS AND PARTITIONS
C   A SPACE OF DIMENSION M INTO N+1 STATISTICALLY EQUIVALENT
C   BLOCKS BY RECORDING BLOCK COORDINATE RANGES IN A MATRIX
C   BLOCK BY THE USE OF SPHERICAL OR POLAR COORDINATESAS
C   AS CUTTING FUNCTIONS.  THE CUTTING COORDINATE USED AT
C   EACH STEP IS CONTAINED IN A VECTOR IRAD.
C.................................................................
C
      SUBROUTINE BLOCKS(VECT,N,NN,M,MM,IRAD,BLOCK,WKVEC)
      DIMENSION VECT(N,M),BLOCK(NN,6),IRAD(N),WKVEC(6)
      ZL=1.0E-8
      BLOCK(1,1)=0.0
      BLOCK(1,2)=1000.
      BLOCK(1,3)=0.0
      BLOCK(1,4)=6.2831853
      BLOCK(1,5)=0.0
      BLOCK(1,6)=3.1415927
      DO 100 J=1,N
      TEMP=0.0
      DO 110 I=1,M
110   TEMP=TEMP+VECT(J,I)**2
      RAD=TEMP**.5
      IF(RAD.GT.ZL)GO TO 112
      TDEG=6.2831853
      PDEG=3.1415927
      GO TO 111
112   TARG=VECT(J,1)/((VECT(J,1)**2+VECT(J,2)**2)**.5)
      PARG=VECT(J,M)/RAD
      IF(TARG.LT.-1.)TARG=-1.
      IF(TARG.LT.1.)GO TO 1122
      TARG=1.0
      IF(PARG.LT.-1.)PARG=-1.
1122  IF(PARG.LT.1.0)GO TO 1123
      PARG=1.0
1123  DEG=ACOS(TARG)
      IF(VECT(J,2).GT.0.)GO TO 111
      IF(DEG.LT.1.5707963)GO TO 113
      DEG=DEG+1.5707963
      GO TO 111
113   DEG=DEG+4.712389
111   CONTINUE
      PDEG=ACOS(PARG)
      WKVEC(1)=TEMP
```

```
      WKVEC(2)=DEG
      WKVEC(3)=PDEG
      DO 120 I=1,NN
      IF(WKVEC(1).GT.BLOCK(I,2))GO TO 120
      IF(WKVEC(1).LT.BLOCK(I,1))GO TO 120
      IF(WKVEC(2).LT.BLOCK(I,3))GO TO 120
      IF(WKVEC(2).GT.BLCCK(I,4))GO TO 120
      IF(M.EQ.2)GO TC 119
      IF(WKVEC(3).LT.BLOCK(I,5))GO TO 120
      IF(WKVEC(3).GT.BLCCK(I,6))GO TO 120
119   CONTINUE
      IBLOCK=I
      GO TO 150
120   CCNTINUE
150   JJ=IRAD(J)
      BLIM=WKVEC(JJ)
      DO 160 I=1,MM
160   BLCCK(J+1,I)=ELOCK(IBLOCK,I)
      BLCCK(IBLOCK,2*JJ)=BLIM
      BLCCK(J+1,2*JJ-1)=BLIM
100   CONTINUE
      RETURN
      END
C...........................................................................
C            SUBROUTINE FCUTZ
C
C     PURPOSE: TO CCMPUTE THE FOUTZ STATISTIC FROM THE
C              BLCCKS DETERMINED BY SUBROUTINE BLOCKS
C              METHCD USES IMSL ROUTINE MDCH TO EVALUATE
C              CHI-SQUARE PROBABILITIES TO EVALUATE THE
C              CLOSED FORM EXPRESSION GIVEN FOR D. THE FN
C              STATISTIC IS GENERATED BY FCUTZ'S CLCSED
C              COMPUTATIONAL FORMULA.
C...........................................................................
C
      SUBROUTINE FOUTZ(BLOCK,NN,MM,FN,M)
      DIMENSION BLOCK(NN,6),P(51)
      CF=FLOAT(M)
      TP=0.0
      DO 100 I=1,NN
      CALL MDCH(BLOCK(I,1),DF,P1,IER)
      CALL MDCH(BLOCK(I,2),DF,P2,IER)
      P3=P2-P1
      P4=(BLOCK(I,4)-BLOCK(I,3))/6.2831853
      IF(M.EQ.2)GO TC 85
      P5=(COS(BLOCK(I,5))-COS(BLOCK(I,6)))/2.0
      GC TO 86
85    P5=1.0
86    P(I)=P3*P4*P5
      TP=TP+P(I)
100   CONTINUE
      FN=0.0
      DO 300 I=1,NN
      AMAX=1.0/NN-P(I)
      IF (AMAX.LT.0.)GO TO 300
      FN=FN+AMAX
300   CONTINUE
      RETURN
      END
C...........................................................................
C             SUBROUTINE BLOCKR
C
C   PURPCSE:
C   THIS SUBROUTINE TAKES N M-VARIATE VECTCRS AND PARTITIONS
C   A SPACE CF CIMENSION M INTC N+1 STATISTICALLY EQUIVALENT
C   BLCCKS BY RECCRCING BLCCK COORDINATE RANGES IN A MATRIX
C   BLCCK BY THE USE OF RECTANGULAR COCRDINATES
```

```
C   AS CUTTING FUNCTIONS.  THE CUTTING COORCINATE USED AT
C   EACH STEP IS CONTAINED IN A VECTOR IRAD.
C.........................................................
C
      SUBROUTINE BLCCKR(VECT,N,NN,M,MM,IRAD,BLOCK,WKVEC)
      DIMENSICN VECT(N,M),BLOCK(NN,MM),IRAD(N),WKVEC(M)
      DO 10 I=1,MM,2
         BLOCK(1,I)=-1000.
         BLCCK(1,I+1)=1000.
10    CONTINUE
      DO 100 J=1,N
         DO 110 I=1,M
110         WKVEC(I)=VECT(J,I)
         DO 120 I=1,NN
            DO 130 II=1,M
               IF(WKVEC(II).LT.BLOCK(I,2*II-1))GO TO 120
               IF(WKVEC(II).GT.BLOCK(I,2*II))GO TO 120
130         CONTINLE
            IBLOCK=I
            GO TO 150
120      CONTINUE
150      JJ=IRAD(J)
         BLIM=WKVEC(JJ)
         DO 160 I=1,MM
            BLCCK(J+1,I)=BLOCK(IBLOCK,I)
160      CONTINUE
         BLOCK(IBLOCK,2*JJ)=BLIM
         BLOCK(J+1,2*JJ-1)=BLIM
100   CONTINUE
      RETURN
      END
C.........................................................
C                         SUBRCUTINE FOUTR
C
C              PURPCSE:
C              TO CCMPUTE THE FOUTZ STATISTIC FROM THE
C              BLOCKS DETERMINED BY SUBROUTINE BLOCKR
C              METHCD USES IMSL ROUTINE MDNCR TO EVALUATE
C              NORMAL PRCBABILITIES TO EVALUATE THE
C              CLOSED FORM EXPRESSION GIVEN FOR D.  THE FN
C              STATISTIC IS GENERATED BY FCUTZ'S CLOSED
C              COMPUTATICNAL FCRMULA.
C.........................................................
C
      SUBROUTINE FOUTR(BLOCK,NN,MM,FN)
      DIMENSICN BLOCK(NN,MM),P(51)
      DO 100 I=1,NN
      P(I)=1.0
      DO 200 J=1,MM,2
      CALL MDNCR(BLCCK(I,J),P1)
      CALL MDNOR(BLCCK(I,J+1),P2)
      P3=ABS(P2-P1)
      P(I)=P(I)*P3
      IF(J.NE.MM-1)GC TO 200
200   CONTINUE
100   CONTINUE
      FN=0.0
      DO 300 I=1,NN
      AMAX=1.0/NN-P(I)
      IF(AMAX.LE.0.0)GO TO 300
      FN=FN+AMAX
300   CONTINUE
      RETURN
      END
```

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